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THE KRONECKER-GAUSSIAN CURVATURE OF HYPERSPACE[*]

BY

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§ 1. *Definition of the Kronecker-Gaussian curvature.*

Let $x', x^2, \cdots, x^{n+1}$ be the coördinates of an euclidean space of $n+1$ dimensions, S_{n+1}, i. e., a space whose arc-element is determined by

$$(1) \qquad ds^2 = \sum_{\lambda=1}^{n+1} (dx^\lambda)^2;$$

then we define any hypersurface, or, as we shall simply say, any space, of n dimensions, R_n, contained in S_{n+1}, by expressing each of the coördinates x^λ in terms of n independent variables $u_1, u_2, \cdots, u_n$.

The arc-element of R_n is given by

$$(2) \qquad ds^2 = \sum_{i,\,k=1}^{n} a_{ik}\,du_i\,du_k,$$

where

$$(3) \qquad a_{ik} = \sum_{\lambda=1}^{n+1} \frac{\partial x^\lambda}{\partial u_i}\,\frac{\partial x^\lambda}{\partial u_k},$$

or

$$(4) \qquad a_{ik} = \sum_{\lambda=1}^{n+1} x_i^\lambda x_k^\lambda,$$

if we agree to indicate differentiation with respect to u_i by the lower index i.

It will be sometimes convenient to write simply

$$\sum(x) \qquad \text{instead of} \qquad \sum_{\lambda=1}^{n+1}(x^\lambda),$$

where (x^λ) stands for any quantity involving or defined by x^λ, e. g.,

$$(5) \qquad a_{ik} = \sum x_i x_k.$$

The $n+1$ direction cosines X^λ of that direction in S_{n+1} which is normal at a point P of R_n to n independent (i. e., not contained in a space of less than n

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dimensions) directions $ds', \cdots, ds^n$ in R_n at P are found to be

$$(6) \qquad X^\lambda = (-1)^{\lambda+1}\beta \begin{vmatrix} x'_1, & \cdots, & x_1^{\lambda-1}, & x_1^{\lambda+1}, & \cdots, & x_1^{n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ x'_n, & \cdots, & x_n^{\lambda-1}, & x_n^{\lambda+1}, & \cdots, & x_n^{n+1} \end{vmatrix},$$

where β denotes the reciprocal square-root of the determinant $|a_{ik}|$. It follows at once that

$$(7) \qquad \sum X^2 = 1,$$

$$(8) \qquad \sum X x_k = 0 \qquad (k = 1, 2, \cdots, n).$$

The Gaussian sphere is defined as the surface (space of n dimensions) whose coördinates are X^λ; its equation is (7).

Let now $d\omega$ be an (n-dimensional) infinitesimal element of R_n and $d\Omega$ the corresponding element of the Gaussian sphere; then the quotient

$$(9) \qquad K = \frac{d\Omega}{d\omega}$$

is KRONECKER's extension of the Gaussian curvature* — the Kronecker-Gaussian curvature of R_n.

To give its analytic expression, we define

$$(10) \qquad \alpha_{ik} = \sum X x_{ik},$$

or

$$(11) \qquad \alpha_{ik} = \beta \begin{vmatrix} x'_{ik}, & x'_1, & \cdots, & x'_n \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ x^{n+1}_{ik}, & x^{n+1}_1, & \cdots, & x^{n+1}_n \end{vmatrix},$$

then

$$(12) \qquad K = \frac{|\alpha_{ik}|}{|a_{ik}|}.$$

§ 2. *Proof that K is expressible in terms of a_i.*

We write symbolically

$$(13) \qquad a_{ik} = f_i f_k = \sum x_i x_k .$$

and introduce the second covariantive derivatives, defined by

$$U^{ik} = U_{ik} - \epsilon f_{ik}(f\phi)(U\phi);$$

then

$$\sum X x^{ik} = \sum X x_{ik} - \epsilon f_{ik}(f\phi) \sum X(x\phi).$$

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* Cf. KILLING, *Die Nicht-Euclidischen Raumformen*, p. 210.

But $\sum X(x\phi) = 0$ on account of (8), so that

$$\sum X x_{ik} = \sum X x^{ik}. \tag{14}$$

From (13) we deduce

$$\sum x_i x^{kl} + \sum x_k x^{il} = f_i f^{kl} + f_k f^{il} = 0$$

on account of I (84).*

Permuting in this equation i, k, l cyclically we obtain two other equations, and from these three equations we have

$$\sum x_i x^{kl} = 0. \tag{15}$$

Solving now the $n + 1$ equations

$$\sum X x^{ik} = \alpha_{ik}, \quad \sum x_1 x^{ik} = 0, \quad \cdots, \quad \sum x_n x^{ik} = 0$$

for the $n + 1$ quantities x^{ik} we obtain

$$x^{ik} = \alpha_{ik} X, \tag{16}$$

$$x^{ir} x^{ks} - x^{is} x^{kr} = (\alpha_{ir}\alpha_{ks} - \alpha_{is}\alpha_{kr}) X^2. \tag{17}$$

Therefore from (7)

$$\alpha_{ir}\alpha_{ks} - \alpha_{is}\alpha_{kr} = \sum (x^{ir} x^{ks} - x^{is} x^{kr}). \tag{18}$$

The sum in (18) can by (15) be transformed into

$$\sum x_k (x^{isr} - x^{irs})$$

which by I (111) combined with

$$\sum x_k (x\phi) = \psi_k (\psi\phi)$$

goes into

$$\epsilon (f^{isr} - f^{irs})(f\phi)\psi_k(\psi\phi).$$

But since

$$\psi_k(\psi\phi)(f\phi) = (n-1)!\, f_k$$

according to I (34) we have, using formula I (113), finally

$$\sum (x^{ir} x^{ks} - x^{rs} x^{kr}) = f^{ir} f^{ks} - f^{is} f^{kr}, \tag{19}$$

$$\alpha_{ir}\alpha_{ks} - \alpha_{is}\alpha_{kr} = f^{ir} f^{ks} - f^{is} f^{kr}, \tag{20}$$

or

$$\alpha_{ir}\alpha_{ks} - \alpha_{is}\alpha_{kr} = (ikrs), \tag{21}$$

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* I quote my paper *A symbolic treatment of the theory of invariants of quadratic differential quantics of n variables*, these Transactions, vol. 4, pp. 441–469, October, 1903, by I, and my paper *Differential parameters of the first order* in this number of the Transactions by D. P.

where $(ikrs)$ is the quadruple index symbol, a quantity which involves the coefficients a_{ik}, its first and second derivatives.*

If now n is even the determinant

$$\Delta = | \alpha_{ik} | \tag{22}$$

can directly be expressed in terms of its minors of the second degree. If n is odd we apply the following general theorems on determinants which can easily be proved

$$\sum_{\lambda,\,\mu} \begin{vmatrix} \alpha_{i\lambda}, & \alpha_{i\mu} \\ \alpha_{k\lambda}, & \alpha_{k\mu} \end{vmatrix} \cdot \begin{vmatrix} A_{i\lambda}, & A_{i\mu} \\ A_{k\lambda}, & A_{k\mu} \end{vmatrix} = \Delta^2, \tag{23}$$

where A_{ik} denotes the minor of α_{ik} in the determinant Δ. Since every A_{ik} is again of even degree, we see that Δ^2 can be expressed in terms of the second degree minors of Δ. It follows then in general from (12) the well known † theorem that

The Kronecker-Gaussian curvature can be expressed in terms of the coefficients a_{ik}, their first and second derivatives.

I mention in passing an interesting result holding in the case where n is odd which follows at once by means of the preceding results from the general formula

$$\alpha_{11} \Delta^{n-1} = \begin{vmatrix} A_{22}, & \cdots, & A_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ A_{n2}, & \cdots, & A_{nn} \end{vmatrix}. \tag{24}$$

We see here that α_{11} itself can be expressed in terms of the coefficients a_{ik}. Therefore, if we call in analogy with the familiar case $n = 2$, $\sum a_{ik} du_i du_k$ the first, and $\sum \alpha_{ik} du_i du_k$ the second fundamental differential quantic, we have the theorem:

If n is odd, the coefficients of the second differential quantic are individually expressible in terms of those of the first differential quantic. The second differential quantic is determined by the first one.

§ 3. *The expression of K in terms of a_{ik}.*

We proceed to compute the determinant of even degree

$$\Delta_{\substack{i_1 i_2 \ldots i_{2m} \\ k_1 k_2 \ldots k_{2m}}} = \begin{vmatrix} \alpha_{i_1 k_1} & \cdots & \alpha_{i_1 k_{2m}} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \alpha_{i_{2m} k_1} & \cdots & \alpha_{i_{2m} k_{2m}} \end{vmatrix}. \tag{26}$$

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* Its unsymbolic expression is given, e. g., in BIANCHI's *Vorlesungen über Differential-Geometrie*, p. 51.

† For references on the (older) theory of the Kronecker-Gaussian curvature see KILLING, l. c., p. 263, 264.

Combining (21) with I (119) we have

$$(27)\qquad \begin{vmatrix} \alpha_{i_1k_1}, & \alpha_{i_1k_2} \\ \alpha_{i_2k_1}, & \alpha_{i_2k_2} \end{vmatrix} = (i_1\,i_2k_1\,k_2) = \epsilon f'_{i_1} f^{2}_{i_2} \begin{vmatrix} (fa)'_{k_1}, & (fa)'_{k_2} \\ (fa)^2_{k_1}, & (fa)^2_{k2} \end{vmatrix},$$

and therefore

$$\Delta_{\substack{i_1 i_2 \ldots i_m \\ k_1 k_2 \ldots k_m}} = \epsilon^m f'_{i_1} f^2_{i_2} \cdots f^{2m}_{i_{2m}} \begin{vmatrix} (fa)'_{k_1} & \cdots & (fa)'_{k_{2m}} \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot \\ (fa)^{2m}_{k_1} & \cdots & (fa)^{2m}_{k_{2m}} \end{vmatrix},$$

with the understanding that the $n-1$ symbols a appearing in every successive pair $(fa)^{2\lambda-1}$ and $(fa)^{2\lambda}$ are equal, otherwise distinct.

The same determinant could have been computed by starting from any other permutation of the rows i with respective change of sign. Adding together all the expressions so obtained and dividing by $(2m)!\,\beta^2$ we have, by finally changing rows and columns,

$$(28)\qquad \beta^2\Delta_{\substack{i_1 \ldots i_m \\ k_1 \ldots k_m}} = \frac{\epsilon^m}{(2m)!}\left((fa)'_{i_1}(fa)^2_{i_2}\cdots(fa)^{2m}_{i_{2m}}\right)\left(f'_{k_1}f^2_{k_2}\cdots f^{2m}_{k_{2m}}\right).$$

If now n is even, we have at once the desired result:

$$(29)\qquad K = \frac{1}{n!\,[(n-1)!]^{n/2}}\left((fa)'(fa)^2\cdots(fa)^n\right)(f).$$

The case where n is odd presents greater difficulties. In this case we have to compute Δ^2 given by formula (23). The degree of the determinant

$$A_{i\lambda} = (-1)^{i+\lambda} \begin{vmatrix} \alpha_{11}, & \cdots, & \alpha_{1,\lambda-1}, & \alpha_{1,\lambda+1}, & \cdots, & \alpha_{1n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{i-1,1}, & \cdots, & \alpha_{i-1,\lambda-1}, & \alpha_{i-1,\lambda+1}, & \cdots, & \alpha_{i-1,n} \\ \alpha_{i+1,1}, & \cdots, & \alpha_{i+1,\lambda-1}, & \alpha_{i+1,\lambda+1}, & \cdots, & \alpha_{i+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \alpha_{n1}, & \cdots, & \alpha_{n,\lambda-1}, & \alpha_{n,\lambda+2}, & \cdots, & \alpha_{nn} \end{vmatrix}$$

being even we can apply (28) and obtain, putting $n = 2m+1$ and indicating each determinant by its diagonal term,

$$A_{i\lambda} = (-1)^{i+\lambda}\frac{\epsilon^m}{(2m)!}\left|(fa)^2_1, (fa)^3_2, \cdots, (fa)^i_{i-1}, (fa)^{i+1}_{i+1}, \cdots, (fa)^n_n\right|$$
$$\times\left|f^2_1, f^3_2, \cdots, f^\lambda_{\lambda-1}, f^{\lambda+1}_{\lambda+1}, \cdots, f^n_n\right|$$

and similar expressions for $A_{k\lambda}$, $A_{i\mu}$ and $A_{k\mu}$.

By taking according to I (119)

$$\begin{vmatrix} \alpha_{i\lambda}, & \alpha_{i\mu} \\ \alpha_{k\lambda}, & \alpha_{k\mu} \end{vmatrix} = (ik\lambda\mu) = \epsilon(fc)'_i(\phi c)'_k(f'_\lambda \phi'_\mu - f'_\mu \phi'_\lambda)$$

we find

$$\Delta^2 = (-1)^{i+k}\epsilon^{n+2}(fc)'_i\,|(fa)_1^2\cdots(fa)_{i-1}^i(fa)_{i+1}^{i+1}\cdots|\cdot(\phi c)'_k|(\phi a)_1^2\cdots(\phi a)_{k-1}^k(\phi a)_{k+1}^{k+1}\cdots|$$
$$\times \sum_{\lambda,\,\mu}(-1)^{\lambda+\mu}\begin{vmatrix} f'_\lambda, & \phi'_\lambda \\ f'_\mu, & \phi'_\mu \end{vmatrix}\cdot\begin{vmatrix} |f_1^2 f_2^3\cdots f_{\lambda-1}^\lambda f_{\lambda+1}^{\lambda+1}\cdots|, & |\phi_1^2\phi_2^3\cdots\phi_{\lambda-1}^\lambda\phi_{\lambda+1}^{\lambda+1}\cdots| \\ |f_1^2 f_2^3\cdots f_{\mu-1}^\mu f_{\mu+1}^{\mu+1}\cdots|, & |\phi_1^2\phi_2^3\cdots\phi_{\mu-1}^\mu\phi_{\mu+1}^{\mu+1}\cdots| \end{vmatrix}.$$

In this sum all the terms in which $\lambda = \mu$ vanish; we can, therefore, perform the summation first with respect to λ, then with respect to μ and divide the result by 2.

But since

$$\sum_{\lambda=1}^{n}(-1)^{\lambda+1}f'_\lambda|f_1^2 f_2^3\cdots f_{\lambda-1}^\lambda f_{\lambda+1}^{\lambda+1}\cdots f_n^n| = |f'_1 f_2^2\cdots f_n^n| = \frac{1}{\beta}(f)$$

and a similar reduction takes place for the other forms, the sum reduces simply to

$$\frac{1}{\beta^2}[(f'f)(\phi'\phi) - (f'\phi)(\phi'f)].$$

The expression thus obtained for Δ^2 holds for all values of i and k. Taking the sum of all these and dividing by n^2 we have

$$\beta^4\Delta^2 = K^2 = \frac{\epsilon^{n+2}}{n^2}\big((fc)'(fa)^2(fa)^3\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2(\phi b)^3\cdots(\phi b)^n\big)$$
$$\times[(f'f)(\phi'\phi) - (f'\phi)(\phi'f)]. \tag{30}$$

Here again the $n-1$ symbols a in two consecutive terms $(fa)^{2\lambda}$ and $(fa)^{2\lambda+1}$ are equal, likewise the symbols b in $(\phi b)^{2\lambda}$ and $(\phi b)^{2\lambda+1}$.

To obtain a further reduction of the above expression we apply D. P. (1) to the product $(f'f)(\phi'\phi)$.

We have

$$(f'f)(\phi'\phi) = (f'\phi)(\phi'f) + \sum_{k=2}^{n}(f^k\phi)(f'\cdots f^{k-1}\phi'f^{k+1}\cdots f^n).$$

But all the terms of this sum become equal if we multiply by $\big((fc)'(fa)^2\cdots(fa)^n\big)$. For instance in

$$(f^3\phi)(f'f^2\phi^4 f^4\cdots f^n)\big((fc)'(fa)^2(fa)^3\cdots f(a)^n\big)$$

we can permute f^2 with f^3 and also $(fa)^2$ with $(fa)^3$ because the corresponding symbols a are equal.

In

$$(f^4\phi)(f'f^2f^3\phi'f^5\cdots f^n)\big((fc)'(fa)^2(fa)^3\cdots(fa)^n\big)$$

we permute f^2 with f^4, also f^3 with f^5 and the symbols a in $(fa)^2$, $(fa)^3$ with those in $(fa)^4$, $(fa)^5$, i. e., $(fa)^2$ with $(fa)^4$ and $(fa)^3$ with $(fa)^5$. In both cases the two expressions become equal to

$$(f^2\phi)(f'\phi' f^3 \cdots f^n)\big((fc)'(fa)^2(fa)^3 \cdots (fa)^n\big).$$

Thus we have the final result

$$K^2 = \frac{1}{n[(n-1)!]^{n+2}}\big((fc)'(fa)^2(fa)^3\cdots(fa)^n\big)\big((\phi c)'(\phi b)^2(\phi b)^3\cdots(\phi a)^n\big) \times (f'\phi' f)(f^2\phi^2\phi). \tag{31}$$

§ 4. *The Kronecker-Gaussian curvature as invariant of a general differential quantic.*

The $\frac{1}{2}n(n+1)$ quantities a_{ik} considered as functions of $u_1, \cdots, u_n$ are not independent if $n > 2$. On account of (4) there must indeed exist $\frac{1}{2}n(n-1)-1$ relations between them.

Let us, however, consider a differential quantic

$$ds^2 = \sum_{i,k=1}^{n} a_{ik}\, du_i\, du_k \tag{32}$$

where the a_{ik} are unrestricted. If now we form the quantities K (30) or K^2 (31) according as n is even or odd, then these expressions are, as is obvious immediately from their structure, differential invariants of (32).

This invariant K might properly be called the Kroneckerinvariant of the differential quantic (32). *If the arc-element ds defined by* (32) *belongs to a space R_n contained in an euclidean space of $n+1$ dimensions, then K becomes the Kronecker-Gaussian curvature, and if $n = 2$ the Gaussian curvature.*

The expressions (30) and (31), especially (31), can be modified in several ways. I mention only that for $n = 3$ the invariant K^2 is identical (leaving a numerical factor aside) with the invariant K_3 given in I (139).

§ 5. *The Kroneckerinvariant K of a space of λ dimensions R_λ represented as a differential parameter of a space of higher dimensions R_n containing R_λ.*

As in D. P. § 3 we define in a general space R_n of n dimensions whose arc-element is determined by

$$ds^2 = \sum_{r,s=1}^{n} a_{rs}\, dx_r\, dx_s \tag{33}$$

a surface (space) R_λ of λ dimensions by the $n - \lambda$ equations

$$U' = \text{const.}, \quad \cdots, \quad U^{n-\lambda} = \text{const.} \tag{34}$$

We shall first determine the arc-element ds of R_λ in terms of λ independent variables $u_1, \cdots, u_n$ and then form the Kroneckerinvariant K of ds^2.

For that purpose we adjoin, as in D. P. § 3, λ arbitrary functions $V', \cdots, V^\lambda$ with the restriction that the functional determinant

$$(35) \qquad \Delta = | V', \cdots, V^\lambda, U', \cdots, U^{n-\lambda} | \neq 0 .$$

According to D. P. (14) the general arc-element ds in R_λ is then determined by

$$(36) \qquad dx_1 = \sum_{k=1}^{\lambda} \rho^k A^{k1}, \quad \cdots, \quad dx_n = \sum_{k=1}^{\lambda} \rho^k A^{kn},$$

where A^{kr} denotes the minor of the element V^k_r in Δ, and where $\rho', \cdots, \rho^\lambda$ are λ arbitrary parameters.

On the other hand, the space R_λ defined by (34) can also be defined by expressing its coördinates x in terms of λ independent variables $u_1, \cdots, u_n$. If we do this, we have for the differentials dx the expressions

$$(37) \qquad dx_r = \sum_{k=1}^{\lambda} \frac{\partial x_r}{\partial u_k} du_k \qquad (r = 1, 2, \cdots, n).$$

To make the expressions (36) and (37) equal we write first $\rho^k du_k$ instead of ρ^k in (36) so that

$$(38) \qquad dx_r = \sum_{k=1}^{\lambda} \rho^k A^{kr} du_k$$

and we have now to set up the integrability-conditions of (38), i. e., the equations

$$(39) \qquad \frac{\partial(\rho^i A^{ir})}{\partial u_k} = \frac{\partial(\rho^k A^{kr})}{\partial u_i}$$

following from

$$(40) \qquad \frac{\partial x_r}{\partial u_k} = \rho^k A^{kr}.$$

If we denote differentiation with respect to x^k by the lower index k we find, M being any function of x,

$$\frac{\partial M}{\partial u_k} = \sum_{s=1}^{n} M_s \frac{\partial x_s}{\partial u_k},$$

and by (40)

$$(41) \qquad \frac{\partial M}{\partial u_k} = \rho^k \sum_{s=1}^{n} M_s A^{ks},$$

so that (39) becomes

$$(42) \qquad \rho^k \sum_s (\rho^i A^{ir})_s A^{ks} = \rho^i \sum_s (\rho^k A^{kr})_s A^{is}$$

which reduces to

$$(43)\qquad \rho^i\rho^k\sum_s(A^{ir}_s A^{ks} - A^{kr}_s A^{is}) = \rho^i A^{kr}\sum \rho^k_s A^{is} - \rho^k A^{ir}\sum \rho^i_s A^{ks}.$$

This equation holds for $i, k = 1, \cdots, \lambda$ and $r = 1, \cdots, n$.

Keeping now i and k fixed we have n equations before us. Instead of using this system of n equations, say

$$(44)\qquad P_1 = 0, \cdots, P_n = 0,$$

we can use the equivalent system of n equations

$$(45a)\qquad \sum_{r=1}^{n} U^\alpha_r P_r = 0 \qquad (\alpha = 1, \cdots, n-\lambda),$$

$$(45b)\qquad \sum_{r=1}^{n} V^k_r P_r = 0 \qquad (k = 1, \cdots, \lambda),$$

which follows from (44) and from which conversely (44) follows on account of (35).

Considering the signification of the quantities A^{kr}, we reduce equation (45a) to

$$\sum_{r,s} U^\alpha_r (A^{ir}_s A^{ks} - A^{kr}_s A^{is}) = 0.$$

But this equation is identically true on account of the relations which arise from differentiating the two equations

$$\sum U^\alpha_r A^{ir} = 0 \qquad \text{and} \qquad \sum U^\alpha_r A^{kr} = 0$$

with respect to x_s. Thus equations (45a) are satisfied without any further condition. Applying similar reductions to the equations (45b) we obtain

$$\sum_s (\rho^k \Delta_s + \rho^k_s \Delta) A^{is} = 0,$$

i. e.,

$$\sum_s \frac{\partial(\rho^k\Delta)}{\partial x_s} A^{is} = 0,$$

and this gives us by (41)

$$\frac{\partial \rho^k \Delta}{\partial u_i} = 0$$

for every value of i different from k. We have therefore

$$\rho^k = \frac{F(u_k)}{\Delta},$$

and by introduction of

$$F(u_k)\,du_k = du'_k$$

and writing again u instead of u' which amounts to $\rho^k = 1/\Delta$ we can state our result as follows :

If a space R_λ *is given by the equations* $U' = const., \cdots, U^{n-\lambda} = const.$, *then by adjoining* λ *arbitrary functions* $V_1' \cdots V^\lambda$ *which have only to satisfy the condition that the functional determinant*

$$\Delta = | V' \cdots V^\lambda U' \cdots U^{n-\lambda} | \neq 0,$$

the differentials dx *of* R_λ *can be written*

$$dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} A^{kr} du_k \qquad (r = 1, \cdots, n) \tag{46}$$

where A^{kr} *denotes the minor of the element* V^{kr} *in* Δ.

To find the expression for ds^2 in terms of $u_1 \cdots u_n$ we apply the symbolic method, introducing symbols for the differential quantic (33) by putting

$$a_{rs} = f_r f_s.$$

We have then

$$ds^2 = \left(\sum_{r=1}^{n} f_r dx_r \right)^2. \tag{47}$$

To form this expression we deduce from (46), understanding by $p_1, p_2, \cdots p_n$, any n quantities

$$\sum_{r=1}^{n} p_r dx_r = \frac{1}{\Delta} \sum_{k=1}^{\lambda} \sum_{r=1}^{n} p_r A^{kr} du_k,$$

or, performing the summation with respect to r on the right side

$$\sum_{r=1}^{n} p_r dx_r = \frac{1}{\beta\Delta} \sum_{k=1}^{\lambda} (V' \cdots V^{k-1} p V^{k+1} \cdots V^\lambda U) du_i.$$

Hence

$$ds^2 = \frac{1}{\beta^2 \Delta^2} \sum_{i,k=1}^{\lambda} (V' \cdots V^{i-1} f V^{i+1} \cdots V^\lambda U) (V' \cdots V^{k-1} f V^{k+1} \cdots V^\lambda U) du_i du_k. \tag{48}$$

Let us introduce also for ds^2 as given in terms of $u_i \cdots u_\lambda$ symbols by writing

$$ds^2 = \left(\sum_{i=1}^{\lambda} F_i du_i \right)^2; \tag{49}$$

then we have

$$F_i = \frac{1}{\beta\Delta} (V' \cdots V^{i-1} f V^{i+1} \cdots V^\lambda U). \tag{50}$$

We now proceed to compute the Kronecker invariant K of the differential quantic (48) assuming λ to be an *even* number.

In this case we have from (29)

$$\lambda! [(\lambda - 1)!]^{\frac{1}{2}\lambda} \cdot K = G = \big((FA)'(FA)^2 \cdots (FA)^\lambda\big)(F' \cdots F^\lambda), \tag{51}$$

where the invariantive brackets ( ) are to be formed with respect to ds_u^2. As to notation we shall in all doubtful cases use the index u when reference to ds_u^2 is required.

We have from (41)

$$(52)\qquad \frac{\partial M}{\partial u_i} = \frac{1}{\Delta}\sum_{r=1}^{n} M_r A^{ir} = \frac{1}{\beta\Delta}(V' \cdots V^{i-1} M V^{i+1} \cdots V^{\lambda} U)$$

and by D. P. (3)

$$|M' \cdots M^{\lambda}|_u = \frac{1}{\Delta^{\lambda}} |M' \cdots M^{\lambda} U' \cdots U^{n-\lambda}| \cdot | V' \cdots V^{\lambda} U' \cdots U^{n-\lambda}|^{\lambda-1},$$

hence

$$|M' \cdots M^{\lambda}|_u = \frac{1}{\Delta} |M' \cdots M^{\lambda} U' \cdots U^{n-\lambda}|,$$

and

$$(53)\qquad \frac{1}{\beta_u}(M)_u = \frac{1}{\beta\Delta}(M' \cdots M^{\lambda} U).$$

To compute β_u we apply D. P. (3) to (50) and obtain

$$\frac{1}{\beta_u}(F)_u = \frac{1}{\beta\Delta}(f' \cdots f^{\lambda} U)$$

which leads by squaring and considering that $(F)_u^2 = \lambda\,!$ on account of I (17) to the required value of β_u,

$$\beta_u^2 = \frac{\lambda\,!\,\Delta^2}{\beta^2 (f' \cdots f^{\lambda} U)^2},$$

or, denoting the denominator which is differential parameter of ds_x^2, by $\Delta^{\lambda} U$,

$$(54)\qquad (f' \cdots f^{\lambda} U)^2 = \Delta^{\lambda} U,$$

$$(55)\qquad \beta_u = \beta\Delta\sqrt{\frac{\lambda\,!}{\Delta^{\lambda} U}}.$$

Thus we find

$$(56)\qquad (M)_u = \sqrt{\frac{\lambda\,!}{\Delta^{\lambda} U}}(M' \cdots M^{\lambda} U)$$

and

$$(57)\qquad (F)_u = \sqrt{\frac{\lambda\,!}{\Delta^{\lambda} U}}(f' \cdots f^{\lambda} U),$$

so that we have

$$\big((FA)'(FA)^2 \cdots (FA)^{\lambda}\big) = \omega\big(\omega(faU)', \cdots, \omega(faU)^{\lambda}, U\big),$$

where

$$\omega = \sqrt{\frac{\lambda\,!}{\Delta^{\lambda} U}}.$$

By means of D. P. (4) the right side of the above expression becomes

$$\omega^{\lambda+1}\left[(faU)', \cdots, (faU)^{\lambda}, U\right]$$

$$+\,\omega^{\lambda}\sum_{k=1}^{\lambda}(faU)^{k}\big((faU')\cdots(faU)^{k-1}, \omega, (faU)^{k+1}\cdots(faU)^{\lambda}, U\big). \tag{58}$$

To form G we have to multiply this by

$$(F)=\omega(f'\cdots f^{\lambda}U).$$

I wish to show that after this multiplication has been performed each term of the sum $\sum_{k=1}^{k=\lambda} = T_1 + T_2 + \cdots$ vanishes.

For that purpose it is sufficient to consider the first term T_1 which may be written more fully

$$T_1 = (f'a^2\cdots a^{\lambda}U)(f'f^2\cdots f^{\lambda}U)\big(\omega, (f^2a^2\cdots a^{\lambda}U), \cdots\big),$$

or, denoting briefly the terms f', $U'\cdots U^{n-\lambda}$ by V,

$$T_1 = (a^2\cdots a^{\lambda}, V)(f^2\cdots f^{\lambda}, V)\big(\omega, (f^2a^2\cdots a^{\lambda}U), \cdots\big). \tag{59}$$

By means of D. P. (1) we develop

$$\begin{aligned}(a^2\cdots a^{\lambda}V)(f^2\cdots f^{\lambda}V) &= (a^2f^3\cdots f^{\lambda}V)(f^2a^3a^4\cdots a^{\lambda}V)\\ &+(a^3f^3\cdots f^{\lambda}V)(a^2f^2a^4\cdots a^{\lambda}V)\\ &+\cdots\cdots\cdots\cdots\end{aligned}$$

But all the terms of this sum become equal after multiplication with $\big(\omega, (f^2a^2\cdots a^{\lambda}U), \cdots\big)$ as one sees by permuting, e. g., in the second term the two equivalent symbols a^2 and a^3. We have then

$$T_1 = \lambda(f^2a^3\cdots a^{\lambda}V)(a^2f^3\cdots f^{\lambda}V)\big(\omega, (f^2a^2\cdots a^{\lambda}U), \cdots\big).$$

On the other hand by permuting the equivalent symbols f^2 and a^2 in (59) we have

$$T_1 = -\,(f^2a^3\cdots a^{\lambda}V)(a^2f^3\cdots f^{\lambda}V)\big(\omega, (f^2a^2\cdots a^{\lambda}U), \cdots\big),$$

whence $T_1 = 0$. It follows in the same way, that also $T_2 = 0, \cdots$. This leads to

$$G=\omega^{\lambda+2}\big((faU)', \cdots, (faU)^{\lambda}, U\big)(f'\cdots f^{\lambda}U)$$

and to *the final result*

$$K=\frac{\lambda^{\frac{1}{2}\lambda}}{(\Delta^{\lambda}U)^{\frac{1}{2}\lambda+1}}\big((faU)', \cdots, (faU)^{\lambda}, U\big)(f', \cdots, f^{\lambda}U), \tag{60}$$

where

$$(f', \cdots, f^{\lambda}U)=(f', \cdots, f^{\lambda}U', \cdots, U^{n-\lambda}),$$

$$\Delta^{\lambda} U = (f', \cdots, f^{\lambda}, U)^2, \; (faU)' = (f', a^{2'}, \cdots, a^{\lambda'}, U), \text{ etc.}$$

Two sets of symbols $a^2 \cdots a^{\lambda}$ are equal in any two successive brackets $(faU)^{2k+1}$, $(faU)^{2k}$, otherwise distinct. All the symbols are symbols of the differential quantic

$$ds^2 = \sum_{i,\,k=1}^{n} a_{ik}\, dx_i\, dx_k .$$

The principles which lead to this expression of K for even values of λ will doubtless also be sufficient to solve the more complicated problem for the case of odd values of n.

Finally I wish to make an interesting application to the case $\lambda = 2$. Formula (60) gives immediately:

$$(61) \qquad K = \frac{2\big((f\psi U), (\phi\psi U), U\big)(f\phi U)}{(f'\phi' U)^2 \cdot (f''\phi'' U)^2}.$$

Let us first assume also $n = 2$. Then the U's disappear and since

$$(f'\phi')^2 = (f''\phi'')^2 = 2$$

we obtain the ordinary Gaussian curvature

$$(62) \quad K = \tfrac{1}{2}\,[(f\psi)(\phi\psi)](f\phi) = \frac{1}{2(EG - F^2)}\left(2\frac{\partial^2 F}{\partial u\, \partial v} - \frac{\partial^2 E}{\partial v^2} - \frac{\partial^2 G}{\partial u^2}\right) + \cdots.$$

Take now $n = 3$ and let R_3 be the ordinary euclidean space, i. e., $a_{ii} = 1$, $a_{ik} = 0$, if $i \neq k$. Then we have only one function U, and if we write the equation $U =$ const. in the form

$$F(x, y, z) = 0,$$

an easy computation transforms the expression (61) into the familiar form

$$(63) \quad K = -\frac{1}{\left[\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial z}\right)^2\right]^2} \cdot \begin{vmatrix} \frac{\partial^2 F}{\partial x^2}, & \frac{\partial^2 F}{\partial x \partial y}, & \frac{\partial^2 F}{\partial x \partial z}, & \frac{\partial F}{\partial x} \\ \frac{\partial^2 F}{\partial y \partial x}, & \frac{\partial^2 F}{\partial y^2}, & \frac{\partial^2 F}{\partial y \partial z}, & \frac{\partial F}{\partial y} \\ \frac{\partial^2 F}{\partial z \partial x}, & \frac{\partial^2 F}{\partial z \partial y}, & \frac{\partial^2 F}{\partial z^2}, & \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial x}, & \frac{\partial F}{\partial y}, & \frac{\partial F}{\partial z}, & 0 \end{vmatrix},$$

so that one and the same formula (61) involves the two apparently so heterogeneous expressions (62) and (63) of the Gaussian curvature.

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